Finite Temperature Equilibrium Properties of Liquid He³ in Fermi Liquid Theory*

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Landau's Fermi liquid theory is used to obtain expressions for the respective T^3 and T^2 terms in low-temperature expansions of the specific heat and magnetic susceptibility of liquid He³ at a temperature T. Such a calculation is essential toward checking the internal consistency of the Fermi liquid theory above 0.05°K; but previous authors have considered only the zero-temperature limit when working with the Landau parameter, f. Coefficients of the finite-temperature corrections derived here are specific functions of f evaluated on the Fermi surface. Existing experimental data on the pressure dependence of the sound velocity, specific heat, and magnetic susceptibility may be used to assign numerical values to these derivatives under the assumption that f depends only on $|\mathbf{k}-\mathbf{k}'|$. This method of estimation is believed to place an upper limit to the magnitude of the T^3 term in the specific heat; and its resulting value is more than sufficient to account for the specific heat's deviating from linearity below 0.05°K. The strong curvature in the specific heat versus temperature curve above 0.05°K is thus concluded to be consistent with the Fermi liquid theory. The T^2 term in the magnetic susceptibility contains the same function as occurs in the T^3 term for the specific heat plus an additional correction related to derivatives of the spin-dependent part of f. A reasonable amount of accuracy is expected in the numerical estimate of this latter term. Assuming this to be the case, this theory is in accord with the susceptibility data of Meyer *et al.* together with the specific-heat data of Brewer *et al.* or Fairbank *et al.*, but it cannot reconcile the specific heat data of Wheatley *et al.* with any existing susceptibility measurements.

I. INTRODUCTION

A. Preliminary Outline

A VARIETY of experiments[1] have now confirmed that liquid He³ conforms to Landau's Fermi liquid theory[2] at the lowest temperatures. However, since the zero-temperature-limit behavior predicted by Landau begins to manifest itself only below about 0.05°K, the major portion of experimental evidence on liquid He³ has yet to receive adequate interpretation. It is the purpose of the present paper to apply the Fermi liquid model to the specific heat C and magnetic susceptibility χ at "finite" temperatures, i.e., higher than those for which C and χ are, respectively, proportional to and independent of temperature T. In this manner some insight may be gained on the experimental

validity of the Fermi liquid theory above 0.05°K. Although several authors have stated that, because of finite quasiparticle lifetimes, this theory can have little if any meaning above 0.05°K, it has been shown by Balian and deDominicis[3] that, within the limits of perturbation theory, Landau's general theory is valid—at least for equilibrium properties—at arbitrarily high temperatures.

Straightforward application of Fermi statistics is used to obtain the respective T^3 and T^2 terms in C and χ as the lowest order corrections. These terms explicitly contain the familiar $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ (9) related to the quasiparticle forward scattering amplitude, and it is shown (Sec. IV) how the known pressure dependence of C, χ, and the velocity of sound can be used to estimate the particular functions of $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ appearing in the formulas. We thereby show that the strong deviation of the specific heat[4] from proportionality to T

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[1] N. Bernardes and D. F. Brewer, Rev. Mod. Phys. **34**, 190 (1962) [review of work on He³].
[2] (a) L. D. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956), [translation: Soviet Phys.—JETP **3**, 920 (1957)]. (b) A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

[3] R. Balian and C. deDominicis, Nucl. Phys. **16**, 502 (1960); C. deDominicis, Physica **26**, *S* 94 (1960).
[4] (a) A. C. Anderson, G. L. Salinger, W. A. Steyert, and J. C. Wheatley, Phys. Rev. Letters **7**, 295 (1961); (b) A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. **130**, 495 (1963).

above 0.05°K may be explained entirely within the framework of the Fermi liquid theory and without having to introduce any adjustable parameters. The large curvature in the specific heat versus temperature curve appears to be completely consistent with the Fermi liquid theory when account is taken of the strong variation of $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ with respect to k and k' as inferred from the pressure dependence of the sound velocity.

For what appears to be a reasonable choice of parameters, however, the results of this calculation indicate that the magnetic susceptibility and specific heat should deviate from their zero-temperature-limit behavior at close to the same temperature. Since the susceptibility remains practically constant up to about 0.09°K,[5,6] this theory seems to be inconsistent with results of Wheatley and his group[4] which show the specific heat deviating from linearity above 0.05°K but is in accord with the specific-heat measurements of Brewer,[7,8] Fairbank,[9] and co-authors.

Under the following heading application of the Fermi liquid theory at finite temperatures is discussed, paying particular attention to how the methods of this paper differ from other approaches. Sections II and III are devoted to calculations of the specific heat and magnetic susceptibility, respectively, and interpretation of the resulting functions of $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ is discussed in Sec. IV. Sections V and VI are, respectively, devoted to comparison with experiment and a discussion of the results.

B. Fermi Liquid Theory

Fundamental assumptions of the Fermi liquid theory are that the quasiparticle excitations of the liquid are spin $\frac{1}{2}$ fermions, that their number, N, is equal to the number of He^3 atoms in the system, and that the state of a given quasiparticle is completely specified by its momentum $\mathbf{k}$ and spin σ. Thus, the entropy S, for a given distribution of quasiparticles is given by

$$S = -\kappa \operatorname{Tr}_\sigma \sum_{\mathbf{k}} \{ n(\mathbf{k},\sigma)\ln n(\mathbf{k},\sigma)$$
$$+ [1 - n(\mathbf{k},\sigma)]\ln[1 - n(\mathbf{k},\sigma)] \}, \quad (1)$$

together with the constraint

$$\operatorname{Tr}_\sigma \sum_{\mathbf{k}} n(\mathbf{k},\sigma) = N. \quad (2)$$

Here $n(\mathbf{k},\sigma)$ is the occupation number of the state

$|\mathbf{k},\sigma\rangle$ and κ is Boltzmann's constant. The energy $\epsilon(\mathbf{k},\sigma)$ of a quasiparticle is defined in terms of the change in total energy E with respect to a variation $\delta n(\mathbf{k},\sigma)$

$$\delta E = \operatorname{Tr}_\sigma \sum_{\mathbf{k}} \epsilon(\mathbf{k},\sigma)\delta n(\mathbf{k},\sigma). \quad (3)$$

With $\epsilon(\mathbf{k},\sigma)$ defined as in (3), it follows, upon minimization of the free energy, $F = E - TS$, subject to (2), that $n(\mathbf{k},\sigma)$ in thermal equilibrium is just the Fermi distribution

$$n(\mathbf{k},\sigma) = \{\exp\beta[\epsilon(\mathbf{k},\sigma) - \mu] + 1\}^{-1}, \quad (4)$$

$(\beta \equiv 1/\kappa T)$ in which μ is the chemical (Gibbs) potential per particle.

Use of (4) in (1) enables one in principle to calculate the entropy as a function of temperature, and consequently the specific heat, for a given quasiparticle energy spectrum. It is well known that the low temperature (linear) specific heat can be described in terms of an effective mass m^* defined by

$$\left(\frac{d}{dk}\epsilon_k^0\right)_{k=k_F} \equiv \frac{k_F}{m^*} \quad (5)$$

where $\epsilon(\mathbf{k},\sigma)$ is assumed to depend only on $k = |\mathbf{k}|$, is independent of σ in the absence of a magnetic field, and at $T = 0$ is given by ϵ_k^0, the zero-temperature limit of ϵ_k. The Fermi momentum k_F (units are used in which $\hbar = 1$) is given by the familiar

$$(4\pi/3)(k_F/2\pi)^3 = N/2V, \quad (6)$$

for N particles of spin $\frac{1}{2}$ in a volume V.

More detailed information as to the nature of ϵ_k is needed to compute the specific heat at higher temperatures. A natural first choice is to let

$$\epsilon_k = \epsilon_k^0 = \frac{k^2 - k_F^2}{2m^*} + \mu_0, \quad (7)$$

where μ_0 is the chemical potential at $T = 0$, i.e., to assume ϵ_k to be independent of temperature and quadratic in k. Thence, one immediately concludes the specific heat to be the same as that for an ideal gas with mass m^*. Experimental evidence, however, clearly eliminates (7) as a plausible model. Recent measurements[4] give for the ratio of m^* to the mass m of a He^3 atom, $m^*/m \approx 2.8$ from the linear portion of the specific heat curve. This corresponds to an equivalent degeneracy temperature,

$$\kappa T_F^* \equiv k_F^2/2m^*, \quad (8)$$

of about 1.78°K. If liquid He^3 then behaved as an ideal gas with a degeneracy temperature of 1.78°K, the specific heat[10] would deviate from being linear by only about 4% at 0.2°K, whereas the deviation is, in fact, greater than 40% at this temperature.

[5] A. C. Anderson, H. Hart, and J. C. Wheatley, Phys. Rev. Letters **5**, 133 (1960).

[6] A. L. Thomson, H. Meyer, and E. D. Adams, Phys. Rev. **128**, 509 (1962).

[7] D. F. Brewer, J. G. Daunt, and A. K. Sreedhar, Phys. Rev. **115**, 836 (1959).

[8] D. F. Brewer and J. R. G. Keyston, Nature, **191**, 1261 (1961); in Proceedings of the Eighth International Conference on Low-Temperature Physics, London, 1962 (to be published).

[9] M. Strongin, G. O. Zimmerman, and H. A. Fairbank, Phys. Rev. **128**, 1983 (1962).

[10] (a) E. C. Stoner, Phil. Mag. **21**, 145 (1936); (b) Proc. Roy. Soc. (London) **A154**, 656 (1936).

Abrikosov and Khalatnikov[11] calculated equilibrium properties at higher temperatures by assuming ϵ_k to be independent of temperature but to depend on k as $(k-k_0)^2$ where k_0 is arbitrary. More recently, Emery[12] has used the concept of a temperature-dependent effective mass in which ϵ_k is given by (7) but with m^* now a function of T. These methods have the advantage that, because of the quadratic ϵ_k versus k relation, quantities such as the specific heat may be computed for the entire temperature range by using standard (tabulated) integrals over the Fermi distribution.[13] They have the disadvantage, however, of not being *explicitly* related to the Fermi liquid theory, by which the following is meant:

Landau has given a prescription for computing single-particle energies at a given temperature,

$$\epsilon(\mathbf{k},\sigma) = \bar{\epsilon}_k + V^{-1}\, \mathrm{Tr}_{\sigma'} \sum_{\mathbf{k}'} f(\mathbf{k},\sigma,\mathbf{k}',\sigma')\delta n(\mathbf{k},\sigma). \quad (9)$$

Here $\bar{\epsilon}_k$ is the quasiparticle energy when the fluid is in total equilibrium, i.e., at zero temperature and pressure and under the influence of no external fields. It differs from $\epsilon_k{}^0$ which, as used in (5), is the single-particle energy at zero temperature and some given, not necessarily zero, pressure. Because of interactions, $\epsilon(\mathbf{k},\sigma)$ depends on the distribution of other quasiparticles in the system and, thus, changes from $\bar{\epsilon}_k$ when the quasiparticle distribution changes from equilibrium by an amount $\delta n(\mathbf{k},\sigma)$. The factor V^{-1} is introduced in (9) to make $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ identical to that used by Landau; if $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ is independent of density, then the second term in (9) will vary linearly with density. It is evident from (9) that $\epsilon(\mathbf{k},\sigma)$ is, in general, a function of temperature and that $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ must enter into expressions for the temperature dependence of various quantities. Thus, it is difficult to see how the Abrikosov-Khalatnikov model of a temperature-independent quasiparticle energy[11] may be interpreted in terms of (9). The concept of a temperature-dependent effective mass appears quite plausible in this context,[14] and it would obviously be desirable to know how (9) may be transformed into an expression such as (7) with m^* a function of T.

In the present paper, the Fermi liquid theory is used to obtain temperature dependences explicitly related to $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$. By inserting the quasiparticle energy (9) in (4), an expansion in powers of T may be derived for a given property of interest in which the coefficients are certain functions of $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ eval-

uated on the Fermi surface. Little, if anything, can be said about the convergence of such a series; so we are still restricted to very low temperatures, and only the lowest order corrections—T^3 and T^2 terms in C and χ, respectively, are treated. This method, thus, has an obvious shortcoming when compared with the temperature-dependent effective-mass approach. But since $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ is the fundamental parameter of the Fermi liquid theory, it is desirable to see how it affects the specific heat and susceptibility in the region just above 0.05°K where deviations from the zero-temperature limit exist, and, indeed, to see whether it is at all possible to explain finite-temperature equilibrium properties in terms of reasonable values for $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$.

II. SPECIFIC HEAT

The specific heat will now be calculated from the relation

$$C = T(\partial S/\partial T)_V. \quad (10)$$

It is thus necessary to obtain an expression for the entropy valid to order T^3 in order to compute the T^3 term in the specific heat. We assume no spin dependence in this part of the calculation so that the index σ need not be included in the notation; and $\epsilon(\mathbf{k})$ and $n(\mathbf{k})$ are rewritten as ϵ_k and n_k, respectively, in which $\epsilon(\mathbf{k})$ is assumed to depend only on k, the magnitude of $\mathbf{k}$.

According to (9), ϵ_k may be expressed as

$$\epsilon_k = \bar{\epsilon}_k + \delta\epsilon_k,$$

where $\delta\epsilon_k$, in the absence of a magnetic field, is the change in ϵ_k that occurs as a result of either raising the temperature or compressing the fluid or a combination of both. But for a calculation of the temperature dependence of the entropy at constant density, as is of interest here, only changes $\delta\epsilon_k$ due to temperature are important. Hence, it is more convenient to write

$$\epsilon_k = \epsilon_k{}^0 + \delta\epsilon_k, \quad (11)$$

in which $\epsilon_k{}^0$, as used in (5), is the quasiparticle energy at zero temperature for a given density ($\epsilon_k{}^0 = \bar{\epsilon}_k$ at zero pressure) and

$$\delta\epsilon_k = 2V^{-1}\sum_{\mathbf{k}'} f_{\mathbf{k}\mathbf{k}'}\delta n_{k'}, \quad (12)$$

where $f_{\mathbf{k}\mathbf{k}'}$ is the spin-independent part of $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ as defined by Landau[2] and $\delta n_{k'}$ now refers only to changes in the distribution function brought about by changing the temperature at constant density. Using (11), an expression for the T^3 term in S can thus be derived at any pressure.

If $Y(\epsilon_k)$ is defined by

$$Y(\epsilon_k) \equiv n_k \ln n_k + (1-n_k)\ln(1-n_k) \quad (13)$$

and an expansion carried out in powers of $\delta\epsilon_k$, Eq. (1)

[11] A. A. Abrikosov and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **32**, 1084 (1957) [translation: Soviet Physics—JETP **5**, 745 (1957)].

[12] V. J. Emery (private communication).

[13] J. McDougall and E. C. Stoner, Phil. Trans. Roy. Soc., London, Ser. A: **237**, 67 (1938).

[14] T. Usui [Phys. Rev. **114**, 21 (1959)] has also criticized the Abrikosov-Khalatnikov spectrum and suggested using a temperature-dependent m^* in connection with data on the thermal expansion coefficient.

may be written as

$$S = -2\kappa V \int_\alpha^\infty \rho(\epsilon_k^0) Y(\epsilon_k^0) d\epsilon_k^0$$

$$- 2\kappa V \int_\alpha^\infty \rho(\epsilon_k^0) \delta\epsilon_k Y'(\epsilon_k^0) d\epsilon_k^0, \quad (14)$$

where the summation has been replaced by an integration in the usual way,

$$\sum_\mathbf{k} \to (V/8\pi^3) \int d\mathbf{k},$$

and[15]

$$\rho(\epsilon_k^0) \equiv \frac{k^2}{2\pi^2} \frac{dk}{d\epsilon_k^0}, \quad (15)$$

and the lower limit α is the value of ϵ_k^0 when $k=0$. Integrating the first term in (14) by parts and noting that

$$Y'(\epsilon_k^0) = -\beta(\epsilon_k^0 - \mu) n'(\epsilon_k^0), \quad (16)$$

then gives

$$S = -2\kappa V \beta \int_\alpha^\infty [P(\epsilon_k^0) - \rho(\epsilon_k^0) \delta\epsilon_k](\epsilon_k^0 - \mu) n'(\epsilon_k^0) d\epsilon_k^0 \quad (17)$$

with

$$P(\epsilon_k^0) \equiv \int_\alpha^{\epsilon_k^0} \rho(\epsilon_{k'}^0) d\epsilon_{k'}^0. \quad (18)$$

The integral appearing in (17) is evaluated by standard techniques.[10,16] In a low temperature expansion (i.e., in powers of β^{-1}) the lower limit may be extended to $-\infty$ and the following general formula established,[10]

$$\int_{-\infty}^\infty (\epsilon_k^0 - \mu)^m n'(\epsilon_k^0) d\epsilon_k^0$$

$$= 0, \quad m \text{ odd integer},$$

$$= \frac{-2m}{\beta^m}(m-1)! c_m, \quad m \text{ even integer} \neq 0,$$

$$= -1, \quad m = 0, \quad (19a)$$

where, in particular,

$$c_2 = \pi^2/12, \quad c_4 = 7\pi^4/720. \quad (19b)$$

Expanding the bracketed term in the integrand of (17) in powers of $\epsilon_k^0 - \mu$ then results in an expression for S in terms of functions evaluated at $\epsilon_k^0 = \mu$. Since μ is itself a function of temperature, an expansion is next carried out in powers of $\mu - \mu_0$, where μ_0 is the value of μ at zero temperature. From the condition (2) on the

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[15] $\rho(\epsilon_k^0)$ is related to $d\tau/d\epsilon$ as used by Landau and others (Ref. 2) by $\rho(\epsilon_k^0) = \frac{1}{2} d\tau/d\epsilon$.
[16] N. F. Mott and H. Jones, *Properties of Metals and Alloys* (Oxford University Press, London, 1936), pp. 175–178.

constancy of the number of particles, we find

$$\mu - \mu_0 = \frac{-2c_2}{\beta^2} \frac{\rho'(\mu_0)}{\rho(\mu_0)} + \delta\epsilon_{k_F}, \quad (20)$$

to the desired degree of accuracy, where $\delta\epsilon_{k_F}$ is $\delta\epsilon_k$ evaluated at $\epsilon_k^0 = \mu_0$, i.e., when $k = k_F$. Using (19) and (20) in (17) then gives the result,

$$S = \frac{8\kappa V c_2}{\beta} \rho(\mu_0) - \frac{16\kappa V}{\beta^3} \left\{ \frac{[\rho'(\mu_0)]^2}{\rho(\mu_0)} c_2^2 - \rho''(\mu_0) c_4 \right\}$$

$$- \frac{8\kappa V c_2}{\beta} \rho(\mu_0) \left(\frac{d\delta\epsilon_k}{d\epsilon_k^0} \right)_{\epsilon_k^0 = \mu_0}, \quad (21)$$

where the first two terms are those which would be obtained for a Fermi gas with energy spectrum $\epsilon_k = \epsilon_k^0$, and the final term represents the effect of ϵ_k being temperature-dependent.

The quantity $\delta\epsilon_k$ is determined in a similar manner, and is, to order T^2,

$$\delta\epsilon_k = \frac{4c_2}{\beta^2} \rho(\mu_0) \left(\frac{\partial \bar{f}_{kk'}}{\partial \epsilon_{k'}^0} \right)_{\epsilon_{k'}^0 = \mu_0}, \quad (22)$$

where

$$\bar{f}_{kk'} \equiv \frac{1}{2} \int_0^\pi f_{\mathbf{k}\mathbf{k}'} \sin\theta d\theta, \quad (23)$$

with θ the angle between $\mathbf{k}$ and $\mathbf{k}'$, it being assumed that $\bar{f}_{kk'}$ is a function only of the magnitudes k and k'.

Using (21) and (22) in (10) gives the desired expression for the specific heat in terms of the Landau parameter $f_{kk'}$. For convenience ϵ_k^0 is now taken to be of the form given by (7), presumably a reasonable approximation near the Fermi surface. One then finds, finally,

$$C = \frac{1}{2} \pi^2 N\kappa \left(\frac{T}{T_F^*} \right) \left[1 - \frac{3\pi^2}{10}(1+\phi) \left(\frac{T}{T_F^*} \right)^2 \right], \quad (24)$$

with

$$\phi \equiv \frac{5Nk_F^2}{8V\kappa T_F^*} \left(\frac{\partial^2 f_{kk'}}{\partial k \partial k'} \right)_{k=k_F, k'=k_F}; \quad (25)$$

and where use has been made of (6), (8), (19b), and the relation (7), $d/d\epsilon_k^0 = (m^*/k) d/dk$. The specific heat resembles that of an ideal Fermi gas with degeneracy temperature T_F^* only if $\phi \ll 1$. It, in fact, turns out that by considering experimental evidence on the pressure dependence of m^* and the sound velocity, ϕ may be expected to be quite large, possibly as big as 70; so C departs from a linear behavior at a much lower temperature than would be predicted for an ideal Fermi gas of mass m^*. The numerical evaluation of ϕ is discussed in Sec. IVA.

[*Note added in proof.* Equation (24) does not include any phonon contribution to the T^3 specific heat, but this omission appears to be of negligible consequence: The

ratio of the T^3 phonon specific heat to the T^3 term in the specific heat of an ideal Fermi gas with degeneracy temperature T_F^* is $16(T_F^*/\theta_D)^3$ where θ_D is the Debye temperature. For particle density and sound velocity appropriate to liquid He³, $\theta_D \approx 15°K$. Since $T_F^* \approx 2°K$, it thus follows that phonons should contribute less than 5% to the T^3 term in (24) even if $\phi = 0$.]

III. MAGNETIC SUSCEPTIBILITY

Choosing a representation in which the quasiparticle spins are individually quantized along the direction of the applied dc magnetic field $\mathbf{H}$, of magnitude H, the energy $\epsilon(\mathbf{k},\sigma)$ may be written as ϵ_k^+ or ϵ_k^- according as to whether the spin is aligned parallel or antiparallel to $\mathbf{H}$ and assuming, once again, $\epsilon_k^\pm$ to be independent of the direction of $\mathbf{k}$. The bulk magnetization of the N particles then has a magnitude M given by

$$M = \gamma \sum_k (n_k^+ - n_k^-), \qquad (26)$$

where γ is the quasiparticle magnetic moment. We assume that M is linear in H to a very good approximation, so only terms of first order in $\epsilon_k^\pm - \epsilon_k$ are important, where ϵ_k (11) is the quasiparticle energy in zero field. Further, eliminating all terms of higher order in temperature than T^2 results in the expression,

$$n_k^+ - n_k^- = -2A_k\gamma H[n'(\epsilon_k^0) + \delta\epsilon_k n''(\epsilon_k^0)], \qquad (27)$$

where

$$\epsilon_k^+ - \epsilon_k^- \equiv -2A_k\gamma H \qquad (28)$$

($A_k = 1$ for the ideal gas), and $\delta\epsilon_k$ is given by (12) or (22). The summation in (26) is next replaced by an integration, the term in $n''(\epsilon_k^0)$ integrated by parts, and, with the aid of (19) and (20) [Eq. (20) remains valid to first order in H], the result,

$$\chi = 2\gamma^2 V\left\{ \rho(\mu_0)A_{k_F} - \frac{2c_2}{\beta^2}\left(\frac{\rho'(\mu_0)}{\rho(\mu_0)}[\rho(\epsilon_k^0)A_k]'_{\epsilon_k^0=\mu_0}\right.\right.$$

$$\left.\left. - [\rho(\epsilon_k^0)A_k]''_{\epsilon_k^0=\mu_0}\right) - \rho(\mu_0)A_{k_F}\left(\frac{d\delta\epsilon_k}{d\epsilon_k^0}\right)_{\epsilon_k^0=\mu_0} \right\}, \qquad (29)$$

established.

For an evaluation of A_k, the complete expression, including the spin-dependent part, for $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ must be used,[2]

$$f(\mathbf{k},\sigma,\mathbf{k}',\sigma') = f_{kk'} + \zeta_{kk'}\mathbf{\sigma}\cdot\mathbf{\sigma}'; \qquad (30)$$

so $\epsilon_k^+ - \epsilon_k^-$ becomes, from insertion of (30) in (9),

$$\epsilon_k^+ - \epsilon_k^- = (2V)^{-1}\sum_{k'} \zeta_{kk'}(n_{k'}^+ - n_{k'}^-) - 2\gamma H, \qquad (31)$$

where the second term comes from interaction with the external field $\mathbf{H}$, which is not included in (9). Substitution of (27) and (28) in (31) results in an equation for A_k which is evidently very similar in form to (29).

With ϵ_k^0 given by (7), the insertion of (28) and (31) in (29) thus gives, after some algebra,

$$\chi = \frac{3N\gamma^2}{2\kappa T_F^*}A_{k_F}^0\left\{1 - \frac{\pi^2}{12}\left(\frac{T}{T_F^*}\right)^2\left[A_{k_F}^0\left(1+\frac{6}{5}\phi\right)-J\right]\right\}, \qquad (32)$$

where A_k^0 is the value of A_k at zero temperature,

$$A_k^0 = [1+\tfrac{1}{2}\rho(\mu_0)\zeta_{k_F,k_F}]^{-1}[1+\tfrac{1}{2}\rho(\mu_0)(\zeta_{k_F,k_F}-\zeta_{k,k_F})], \qquad (33)$$

and where ϕ has been introduced previously (25), and J, defined by

$$J \equiv k_F^2\left(\frac{\partial^2 A_k}{\partial k^2}\right)_{k=k_F} + \frac{k_F^2}{A_{k_F}^0}\left(\frac{\partial A_k^0}{\partial k}\right)^2_{k=k_F}, \qquad (34)$$

is zero unless ζ_{k,k_F} varies with k. [$\zeta_{kk'}$ is the angular average defined precisely as in (23).]

In the case $J=0$, the above result for χ is readily interpreted. First, consider the zero-temperature limit, which reduces to that derived previously[2] and may be written as

$$\chi_0 = \chi_0^*(1 + \chi_0^*\zeta_{k_F,k_F}/4\gamma^2 V)^{-1}, \qquad (35)$$

where

$$\chi_0^* \equiv 3N\gamma^2/2\kappa T_F^* \qquad (36)$$

is the value of χ at $T=0$ in the absence of an exchange interaction ($\zeta_{kk'}=0$). It is then easy to show that if (35) is assumed valid at all temperatures—i.e., the subscripts "0" removed—then, to order T^2, the expression (32) with $J=0$ is just what would be obtained for χ given as in (35) with

$$\chi^* = \chi_0^*\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F^*}\right)^2\left(1+\frac{6}{5}\phi\right)\right], \qquad (37)$$

the value of χ for $\zeta_{kk'}=0$. That (37) is indeed what one would compute for χ in the neglect of $\zeta_{kk'}$ is seen by setting A_k equal to unity in (29).

Abrikosov and Khalatnikov[11] assumed (35) to hold at all temperatures, and Stoner,[10b] in his work on collective electron ferromagnetism, showed such a form to be justifiable in the case where the exchange interaction reduces to an effective temperature-independent Weiss molecular field. The molecular field approximation is formally equivalent to assuming $\zeta_{kk'}$ to be independent of k and k', as well as being temperature-independent; and it is in this instance, and only in this instance, that (35) is valid at all temperatures since, in general, J will be zero only if $\zeta_{kk'}$ is a constant.

IV. NUMERICAL ESTIMATE OF FINITE TEMPERATURE CORRECTIONS

An estimate of the parameters ϕ and J appearing in (24) and (32) is made from experimental results on the pressure dependence of various quantities. In part A of this section, ϕ is shown to be related to the pressure dependence of the velocity of sound and is estimated to

be about 70 at zero pressure. Part B is concerned with calculating J, which turns out to be about 10 at equilibrium density. That J is computed as nearly an order of magnitude less than ϕ stems from the experimental fact that the magnetic susceptibility is much less dependent upon pressure than is the sound velocity.

A. Evaluation of ϕ

The parameter ϕ depends on the manner (25) in which $\bar{f}_{kk'}$ varies with k and k'. Landau and others[2] have shown the relation which exists between $\bar{f}_{k_F,k_F}$, the effective mass m^*, and the velocity of ordinary sound c. It may be expressed as (see Appendix B)

$$\bar{f} = (V/N)(mc^2 - k_F^2/3m^*),\qquad (38)$$

where $\bar{f}$ is an abbreviated notation for $\bar{f}_{k_F,k_F}$. Observation of the variation of c and m^* with pressure together with the known pressure versus density curve[17] for liquid He^3 then enables one to deduce how $\bar{f}$ varies with k_F.

In particular, the quantity $d^2\bar{f}/dk_F^2$ may thus be determined experimentally. This is not, of course, in general sufficient to give any information about $(\partial^2\bar{f}/\partial k\partial k')_{k=k_F,k'=k_F}$, which is involved in the expression for ϕ. However, the minimal assumption that $f_{kk'}$ depends only on the magnitude $|\mathbf{k}-\mathbf{k'}|$, namely,

$$f_{kk'} = f(|\mathbf{k}-\mathbf{k'}|),\qquad (39)$$

is sufficient to establish that

$$\left(\frac{\partial^2 f_{kk'}}{\partial k\partial k'}\right)_{k=k_F;k'=k_F} = \frac{1}{4}\frac{d^2f}{dk_F^2} - \frac{f - f(0)}{2k_F^2},\qquad (40)$$

as shown in Appendix A, where $f(0)$ is the value of $f_{kk'}$ for $\mathbf{k}=\mathbf{k'}$. Hence, apart from the term $f(0)$, ϕ may be directly related to the pressure variation of $\bar{f}$ simply by invoking the reasonable assumption (39). An order of magnitude estimate is given for $f(0)$ in Appendix A, and it is concluded that the second term in (40) is negligible compared with the first; so the appearance of $f(0)$ is of no immediate concern.

We introduce dimensionless units

$$x \equiv k_F/k_{F_0},\qquad (41a)$$

$$y \equiv c^2/c_0^2,\qquad (41b)$$

$$z^* \equiv (k_F^2/m^*)/(k_{F_0}^2/m_0^*) = T_F^*/T_{F_0}^*,\qquad (41c)$$

in which the subscript "0" indicates values taken at zero pressure, as well as zero temperature. Assuming the second term in (40) to be unimportant (Appendix A) and making use of (6), (25), (38), (40), and (41), then,

$$\phi = \frac{5}{32}\left(\frac{mc_0^2}{\kappa T_{F_0}}\right)\frac{m^*}{m}x^3\frac{d^2}{dx^2}\left(\frac{y}{x^3}\right) - \frac{5x^5}{48z^*}\frac{d^2}{dx^2}\left(\frac{z^*}{x^3}\right),\qquad (42)$$

[17] R. H. Sherman and F. J. Edeskuty, Ann. Phys. (N. Y.) **9**, 522 (1960).

where T_{F_0}, the degeneracy temperature for an ideal gas with the same mass and density as liquid He^3 under zero pressure, is approximately 5.0°K. Since $c_0 = 182$ meters/sec,[18] the factor $(mc_0^2/\kappa T_{F_0})$ is 12/5.

The quantities $y(x)$ and $z^*(x)$ may be taken from existing experimental data. Treating z^* first, it is seen, (24) and (41c), that z^* is the inverse of the linear term in the specific heat normalized to its value at zero pressure. Anderson et al.[4] and Brewer and Keyston[8] have measured the pressure dependence of the linear specific heat, and a plot of z^*/x^3 as determined from their data is shown in Fig. 1a. For pressures up to 20 atm, z^*/x^3 is an approximately linear function of x, from which we conclude that the second term in (42) is negligible.

Published results[18] on the variation of c with pressure below 0.1°K are used to obtain the plot of y/x^3 versus x presented in Fig. 1(b). The empirical relation

$$y/x^3 = 2x^{6.4} - x^{-1}\qquad (43)$$

is seen to give quite a good fit to the experimental points over the entire pressure range covered (0.1 atm to 28.9 atm). Since mc^2 is given by the change of pressure with respect to density, the strong curvature evidenced by (43) should not be surprising in view of the highly nonlinear characteristic of the pressure versus density curve[17] for liquid He^3. Equation (43) has been used to compute the required second derivative, giving $\phi = 70.5$ at equilibrium density with $m^*/m = 2.8$. Since both m^*[4,8] and the curvature of y/x^3 (43) increase with pressure, ϕ increases with pressure and becomes about 270 at 26.7 atm ($x = 1.12$) assuming[4,8] m^* to increase by a factor of 1.6 in going from 0 to 26.7 atm pressure.

B. Evaluation of J

Analogous to the work of the preceding section, J is determined from a knowledge of how $\bar{\zeta}$, the abbreviated notation for $\bar{\zeta}_{k_F,k_F}$, varies with pressure. As before, we assume $\zeta_{kk'}$ to depend only on the magnitude $|\mathbf{k}-\mathbf{k'}|$ so that, similar to (40), the pertinent relations,

$$\left(\frac{\partial}{\partial k}\bar{\zeta}_{kk_F}\right)_{k=k_F} = \frac{1}{2}\frac{\partial\bar{\zeta}}{dk_F}\qquad (44a)$$

$$\left(\frac{\partial^2\bar{\zeta}_{kk_F}}{\partial k^2}\right)_{k=k_F} = \frac{1}{4}\frac{d^2\bar{\zeta}}{dk_F^2} + \frac{\bar{\zeta} - \zeta(0)}{2k_F^2},\qquad (44b)$$

are derived in Appendix A, where $\zeta(0)$ is $\zeta_{kk'}$ evaluated at $\mathbf{k}=\mathbf{k'}$.

A common practice[6] is to introduce a quantity T_F^{**} defined as the effective Fermi temperature as given by the low-temperature susceptibility,

$$\chi_0 = 3N\gamma^2/2\kappa T_F^{**}\qquad (45a)$$

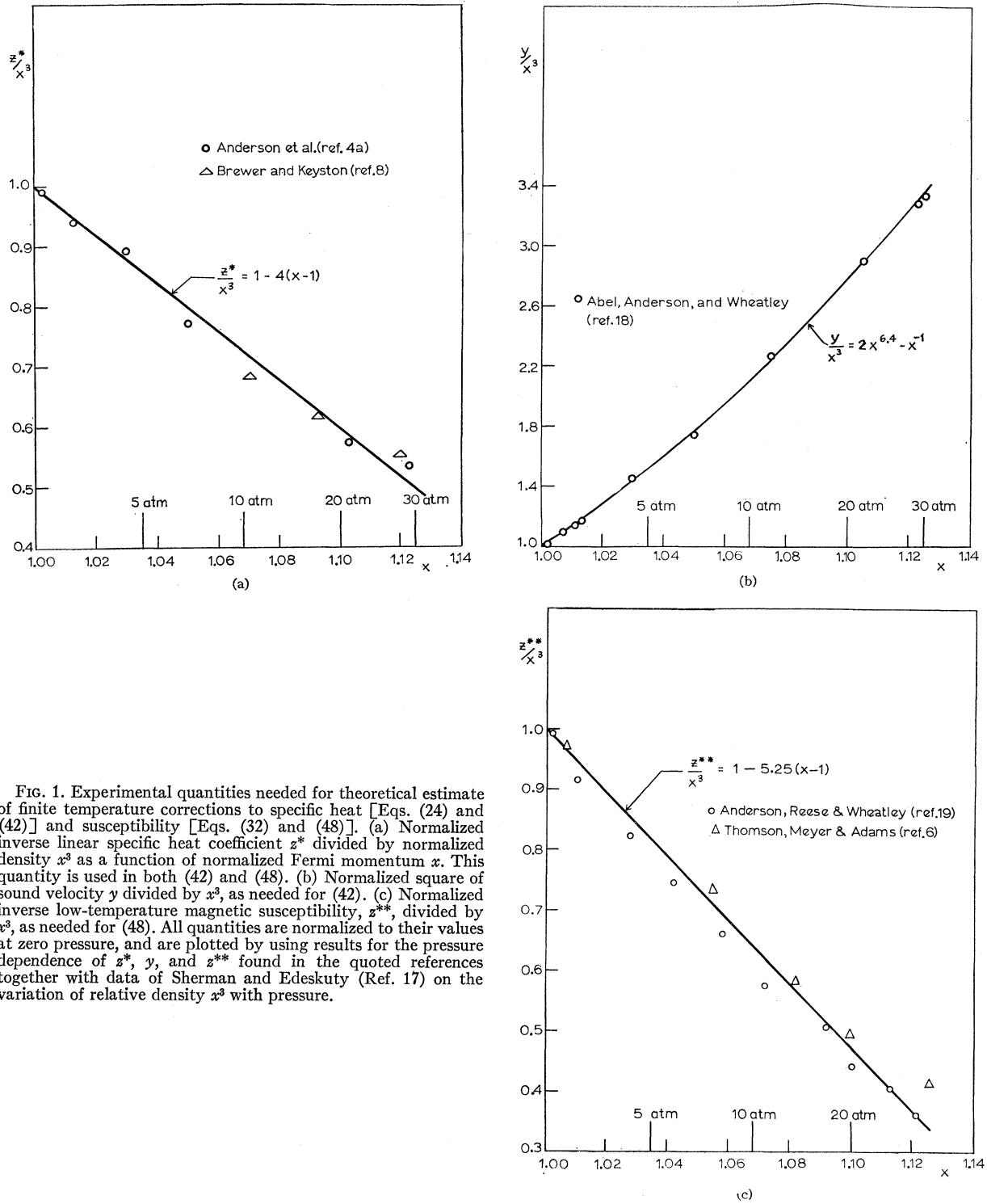[18] W. R. Abel, A. C. Anderson, and J. C. Wheatley, Phys. Rev. Letters **7**, 299 (1961).

FIG. 1. Experimental quantities needed for theoretical estimate of finite temperature corrections to specific heat [Eqs. (24) and (42)] and susceptibility [Eqs. (32) and (48)]. (a) Normalized inverse linear specific heat coefficient z^* divided by normalized density x^3 as a function of normalized Fermi momentum x. This quantity is used in both (42) and (48). (b) Normalized square of sound velocity y divided by x^3, as needed for (42). (c) Normalized inverse low-temperature magnetic susceptibility, z^{**}, divided by x^3, as needed for (48). All quantities are normalized to their values at zero pressure, and are plotted by using results for the pressure dependence of z^*, y, and z^{**} found in the quoted references together with data of Sherman and Edeskuty (Ref. 17) on the variation of relative density x^3 with pressure.

or, equivalently,

$$A \equiv T_F{}^*/T_F{}^{**}, \qquad (45b)$$

where A is an abbreviated notation for $A_{k_F}{}^0$, and, using

(45b) in (33) together with (7),

$$\bar{\zeta} = (8V/3N)\kappa(T_F{}^{**} - T_F{}^*). \qquad (46)$$

Defining

$$z^{**} \equiv T_F{}^{**}/T_{F_0}{}^{**}, \qquad (47)$$

TABLE I. Values of finite-temperature corrections, ϕ, and J, near equilibrium density. The quantities ϕ(exptl) and J(exptl) are from data of the sources shown in first column. Further detail may be found in Sec. V.

Source	m^*/m	A_0	ϕ(exptl)	ϕ(theoret)[e]	J(exptl)	$J(\lambda=0)$[f]	$J(\lambda=1)$[f]
Brewer and Keyston[a]	2.2	$\cdots$	8	55	$\cdots$	$\cdots$	$\cdots$
Wheatley et al.[b]	2.8	$\cdots$	30	70	$\cdots$	$\cdots$	$\cdots$
Brewer and Keyston[a] and Meyer et al.[c]	2.2	5.3	8	55	12	12.0	14.2
Wheatley et al.[b] and Meyer et al.[c]	2.8	4.1	30	70	125	7.6	9.2
Brewer and Keyston[a] and Anderson et al.[d]	2.2	4.4	8	55	34	8.7	10.4
Wheatley et al.[b] and Anderson et al.[d]	2.8	3.4	30	70	122	5.1	6.3

[a] See Ref. 8. [Data of Strongin, Zimmerman, and Fairbank (Ref. 9) give similar results.]
[b] See Ref. 4.
[c] See Ref. 6.
[d] See Ref. 5.
[e] See Eq. (42).
[f] See Eq. (50).

where $T_{F_0}^{**}$ is the value of T_F^{**} at equilibrium density, then results in the following expression for J by use of (33), (41), and (44)–(47) in (34):

$$J = -\frac{x^5}{4z^*z^{**}}\left[\frac{d^2}{dx^2}\left(\frac{z^{**}}{x^3}\right) - A_0\frac{d^2}{dx^2}\left(\frac{z^*}{x^3}\right)\right]$$
$$+\frac{A-1}{2}[1-\bar{\zeta}/\zeta(0)]$$
$$+\frac{x^8}{4z^*z^{**}A_0}\left[\frac{d}{dx}\left(\frac{z^{**}}{x^3}\right) - A_0\frac{d}{dx}\left(\frac{z^*}{x^3}\right)\right]^2, \quad (48)$$

with the first two terms representing the contribution of $(\partial^2 A_k/\partial k^2)_{k=k_F}$ to (34) and the final term that of $(\partial A_k/\partial k)^2_{k=k_F}$. The quantity A_0 is the value of A at equilibrium density.

Anderson, Reese, and Wheatley[19] and Thomson, Meyer, and Adams[6] have measured the pressure dependence of the low-temperature susceptibility; so $z^{**}(x)$ may be inferred from their data. As shown in Fig. 1(c), z^{**}/x^3 appears to vary linearly with x, and since [Fig. 1(a)] z^*/x^3 is also linear in x, we take the first term in (48) to be zero.

In the evaluation of ϕ, $f(0)$ was unimportant because of the very large value of $d^2\bar{f}/dk_F^2$. Here, however, $d^2\bar{\zeta}/dk_F^2$ is zero; so the second term in (48), containing $\zeta(0)$, must be included. It seems unlikely however, (Appendix A) that $|\bar{\zeta}-\zeta(0)|$ is any larger than the order of magnitude of $\zeta(0)$.[20] Introducing a parameter, λ, defined by

$$\lambda \equiv 1-\zeta(0)/\bar{\zeta}, \quad (49)$$

and using the straight-line slopes given in Fig. 1(a) and 1(c) in (48) then gives

$$J = \frac{A_0-1}{2}\lambda + \frac{(4A_0-5.25)^2}{4A_0} \quad (50)$$

[19] A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. 127, 671 (1962).
[20] It is at first tempting to set $\zeta(0) = -4f(0)$ by invoking the exclusion principle to require $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ to be zero for equal momenta and parallel spins. However, this assumption cannot be made since $f(\mathbf{k},\sigma,\mathbf{k}',\sigma')$ is not identically the forward scattering amplitude, which is zero for $\mathbf{k}=\mathbf{k}'$ and $\sigma=\sigma'$. [See Eqs. (A1.14), (A2.14), and (A2.15) of Ref. 2b; see also D. Hone, Phys. Rev. 125, 1494 (1962).]

at equilibrium density, where λ is of undetermined sign but should be no larger in magnitude than the order of unity.

Values of A_0 inferred from experiment are given in Table I, where corresponding values of J are also given for both $\lambda=0$ and $\lambda=1$. It is seen that J, as computed from (50), may be expected to be of the order of 10 and that there is at most a 25% difference between $J(\lambda=0)$ and $J(\lambda=1)$.

V. COMPARISON WITH EXPERIMENT

Experimental results for ϕ and J deduced from the manner in which the specific heat and susceptibility depart from their zero-temperature limits are now examined. Figures 2 and 3 together with Table I contain most of the information pertinent to this section.

Wheatley,[4] Brewer,[7,8] Fairbank,[9] and their co-workers have measured the specific heat of liquid He³. Brewer and Keyston[8] (0.04°K) and Strongin, Zimmerman, and Fairbank[9] (0.054°K) find $m^*/m \approx 2.2$ near zero pressure, the figures in parentheses representing the lowest temperatures reached. Wheatley and his group, however, concluded that $m^*/m \approx 2.8$ in their earlier work[4a] at 0.12 atm down to 0.008°K and more recently,[4b] in measurements down to 0.015°K, find $m^*/m \approx 2.9$. These discrepancies in m^*/m make a large difference in the experimental value to be assigned to ϕ since (Fig. 2) the data of Refs. 8 and 9 indicate that C is linear to within 10% up to about 0.13°K, while Wheatley's results show a departure from linearity by 10% at close to 0.05°K. This means that, choosing ϕ so as to obtain agreement with the lowest temperature departure of the specific heat from linearity, we find, near zero pressure, $\phi \approx 8$ if $m^*/m=2.2$ and $\phi \approx 30$ if $m^*/m = 2.8$. In Sec. IVA, it will be remembered, ϕ was predicted to be 70.5 for $m^*/m=2.8$ and would have been set at 55 for $m^*/m=2.2$ since, by (42), it is linear in m^*/m.

There is similar lack of certainty as to the temperature at which χ ceases to be independent of T, as illustrated in Fig. 3. If χ, normalized to its zero-temperature limit, is approximated by

$$\chi/\chi_0 = 1 - bT^2, \quad (51)$$

with T in °K, then $b \approx 2$ from the data of Anderson, Hart, and Wheatley[5] and $b \approx 7$ from the more recent results of Thomson, Meyer, and Adams.[6] Given b as defined in (51), J may then be computed from (32) provided A, T_F^*, and ϕ are all known. The lower 4 rows of Table I give results of such an analysis, while values of J predicted by (50) for the given A_0 are presented in the last 2 columns of the table.

VI. CONCLUSIONS

The coefficient of the T^3 term in the specific heat, as inferred from the experimental variation of C with T, is significantly less than the "theoretical" value computed from the variation of sound velocity with pressure (Sec. IVA). For $m^*/m = 2.8$, ϕ is a factor of 2.4 smaller than would be computed from (42), while if $m^*/m = 2.2$, Eq. (42) gives a figure too large by a factor of 7. Thus, the Fermi liquid theory together with assumption (39) actually "overexplains" the strong departure of the specific heat from a linear behavior at low temperatures, whereas it might have been supposed that, because its previous results[2] gave the same temperature dependence as an ideal Fermi gas model, Landau's theory would be incapable of producing so marked a curvature in the vicinity of 0.1°K.

That (42) appears to overestimate ϕ is, we feel, encouraging support to the view that the Fermi liquid theory is applicable above the zero-temperature limit. This is because Eq. (42) probably represents an upper limit to ϕ for the following reason: In general $f_{kk'}$ itself, like ϵ_k, will depend upon the distribution of quasiparticles; or, in other words, the total energy E will have functional derivatives of higher order than $\delta^2 E/\delta n_k \delta n_{k'}$. This may be expressed, at zero temperature,
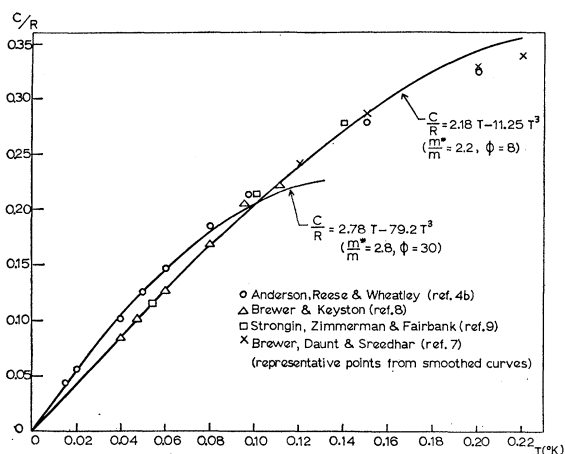

FIG. 2. Specific heat C of liquid He³ divided by gas constant R near zero pressure. Experimental values for the parameter ϕ appearing in Eq. (24) are determined by fitting curves of the form $C/R = cT - dT^3$ to the lowest temperature data, with T in °K. From (24), $c = (\pi^2/2)(m^*/5m)$ and $d = (3\pi^4/20)(m^*/5m)^3(1+\phi)$ since T_{F0}, the degeneracy temperature for an ideal Fermi gas with the same mass and density as liquid He³ at zero pressure, is approximately 5.0°K.
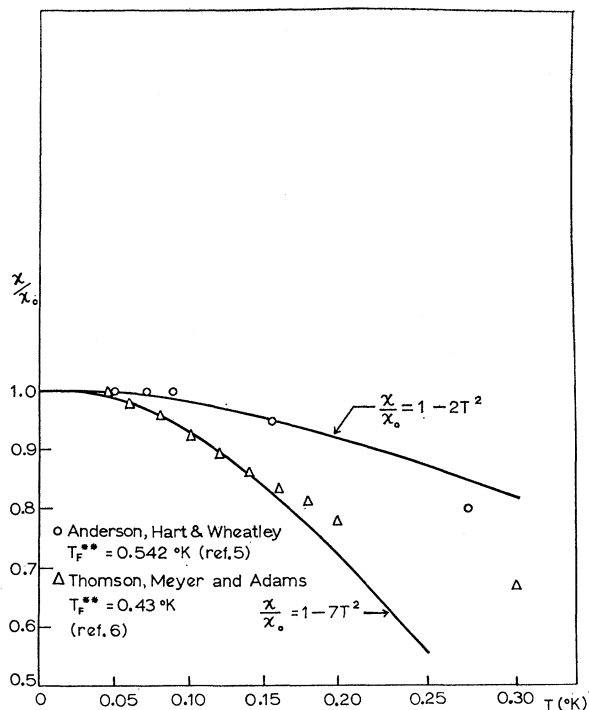

FIG. 3. Magnetic susceptibility χ near equilibrium density, normalized to its zero-temperature limit χ_0. Experimental values for the parameter J appearing in (32) are determined by fitting curves of the form $\chi/\chi_0 = 1 - bT^2$ to the lowest temperature data, with T in °K. From (32),

$$b = (\pi^2/12)(m^*/5m)^2[(5+6\phi)(T_F^{**}m^*/m)^{-1} - J]$$

with $T_{F0} = 5.0$°K and T_F^{**}, the equivalent degeneracy temperature as derived from the low-temperature susceptibility (45), in °K. Knowledge of m^*/m, T_F^{**}, and ϕ is thus required, together with b, to evaluate J (see Table I).

by saying that $f_{kk'}$ depends upon density; so the assumption (39) should have been written as

$$f_{kk'} = f(|\mathbf{k} - \mathbf{k}'| ; k_F) \tag{39a}$$

to allow explicitly for such a density dependence. It is then obvious that (40) is no longer valid since $\bar{f}$ can now vary strongly with respect to k_F without $f_{kk'}$ varying at all with respect to k and k'. Hence if $f_{kk'}$ is an increasing function of density, the quantity $\partial^2 \bar{f}_{kk'}/\partial k \partial k'$ will be less than given by (40) and thus ϕ smaller than estimated by (42).

Density dependence of $f_{kk'}$ is related to the concept of rearrangement energy [21,22] in Brueckner's many-body theory; if there is a rearrangement energy, then $f_{kk'}$ is density-dependent, as shown in Appendix B. In the Brueckner-Gammel theory[23] of liquid He³, the average energy per particle is -0.96κ°K at equilibrium density while the single-particle energy at the Fermi surface is

[21] K. A. Brueckner, Phys. Rev. 110, 597 (1958).
[22] K. A. Brueckner and D. T. Goldman, Phys. Rev. 117, 207 (1960).
[23] K. A. Brueckner and J. L. Gammel, Phys. Rev. 109, 1040 (1958).

−3.61κ°K (quoted in Ref. 21); so the rearrangement energy can be expected to be positive and perhaps a factor of 3 greater than the magnitude of the average energy per particle [which, at equilibrium density, is the single-particle energy as defined in (3)]. Thus, it appears reasonable that $f_{kk'}$ is an appreciably increasing function of density and hence equation (42), which neglects this density dependence, gives an upper limit to ϕ which may overestimate the quantity by a considerable amount. Appendix B contains a more quantitative discussion of this point.

If $f_{kk'}$ is a function of the quasiparticle distribution, then the exchange term, $\zeta_{kk'}$, may likewise exhibit a density dependence, thus, introducing an error in the previous estimate for J. The situation is further complicated by the appearance of additional terms in the expression for χ to order T^2 when $f_{kk'}$ and $\zeta_{kk'}$ depend upon the quasiparticle distribution (Appendix C). However, since $\zeta_{kk'}$ is related to the *difference* between scattering amplitudes for collisions involving quasiparticles with parallel and antiparallel spins, it may be reasonable to suppose that $\zeta_{kk'}$ is very much less dependent upon density than $f_{kk'}$, i.e., that the dependence of scattering amplitude upon k_F is approximately equal for parallel and antiparallel spins. Experimental evidence might tend to support this view since the pressure dependence of $\bar{f}$ is much greater than that of $\bar{\zeta}$. Hence, Eq. (44) is likely to be considerably more valid than (40) and, thus, the estimate for J (50) more reliable than the corresponding one for ϕ (42). Furthermore, if the density dependence of $\zeta_{kk'}$ is negligible compared with that of $f_{kk'}$, then the effect of additional terms in the expression for χ to order T^2 is to make J, as appearing in (32), smaller than defined by (34). (See Appendix C.)

Turning now to the results summarized in Table I, the susceptibility data of Thomson, Meyer, and Adams[6] together with $m^*/m = 2.2$, as taken from specific-heat measurements of Refs. 8 and 9, are seen to be consistent with the ideas expressed above, namely, that ϕ can be considerably less than predicted by (42) but that J should not be much different from the value given by (50). If, however, C departs from linear by 10% at 0.05°K, as indicated by the Illinois results,[4] then the susceptibility as well should cease being constant at that temperature since we would expect J to be negligible in (32) for such a large value of ϕ. Since χ appears to remain constant to within 10% up to at least 0.1°K, the specific heat data of Ref. 4 cannot be correlated with susceptibility data except by choosing J an order of magnitude larger than given by (50)—and such a choice does not appear reasonable within the framework of this calculation.

In conclusion, two points may be stressed. First, the Fermi liquid theory itself explains why the specific heat does not become linear until very low temperatures: The variation of $\bar{f}$ with pressure may be used to predict an upper limit to the magnitude of the coefficient of

the T^3 term in the specific heat, and this upper limit, at equilibrium density, is at least a factor of 2 larger than is needed in order for the C versus T curve to have significant curvature down to 0.05°K. It may also be pointed out that since ϕ increases strongly with pressure according to (42), the temperature at which C becomes linear in T becomes progressively lower at higher pressures, which is in agreement with experiment.[4,8]

Secondly, with what appears to be a reasonable estimate for J, this calculation is able to account for magnetic susceptibility data of Thomson, Meyer, and Adams[6] if $m^*/m \approx 2.2$, as measured by Brewer and Keyston[8] and Strongin, Zimmerman, and Fairbank[9]; but it is unable to make any existing susceptibility measurements consistent with the results of Wheatley and his co-workers[4] that $m^*/m \approx 2.8$.

ACKNOWLEDGMENTS

APPENDIX A

Derivatives of $\bar{f}_{kk'}$

We derive here Eqs. (40) and (44) and discuss other pertinent properties of $f_{kk'}$ based upon assumption (39). Consider any function g of $|\mathbf{k}-\mathbf{k}'|$ which possesses a power series expansion,

$$g(K) = g_0 + \sum_{n=1}^{\infty} g_n K^n, \tag{A1}$$

where

$$K \equiv |\mathbf{k}-\mathbf{k}'| = [k^2 + k'^2 - 2kk'\cos\theta]^{1/2}. \tag{A2}$$

Performing the angular integration indicated by (23), gives at once

$$\bar{g}_{kk'} = g_0 + \frac{1}{2kk'} \sum_{n=1}^{\infty} \frac{g_n}{n+2} [(k+k')^{n+2} - (k-k')^{n+2}], \tag{A3}$$

and

$$\bar{g} = g_0 + 2 \sum_{n=1}^{\infty} \frac{g_n}{n+2} (2k_F)^n \tag{A4}$$

on the Fermi surface.

Assuming the functions suitably behaved, term-by-term differentiation of (A3) and (A4) then leads to the relations (40) and (44) with $\bar{f}_{kk'}$ and $\bar{\zeta}_{kk'}$ substituted for $\bar{g}_{kk'}$.

It is common in the literature to expand $g_{kk'}$ in terms of Legendre polynomials when $\mathbf{k}$ and $\mathbf{k}'$ are both on the Fermi surface

$$g(\theta) = \sum_n G_n P_n(\cos\theta), \tag{A5}$$

where $g(\theta) \equiv g_{kk'}(k=k_F, k'=k_F)$ may be either $f_{kk'}$ or $\zeta_{kk'}$ on the Fermi surface. From the orthogonality of Legendre polynomials, we have

$$\bar{g} = G_0, \tag{A6}$$

and further, since[24] $P_n(1) = 1$,

$$g_0 = \sum_n G_n. \tag{A7}$$

Thus if the first few terms of (A5) give a good approximation to $g(\theta)$, then it is reasonable to assume g_0 is of the same order of magnitude as $\bar{g}$. This assumption has been made in all previous calculations and is made here in order to assert that the second term in (40) is negligible compared with the large value of $d^2\bar{f}/dk_F^2$ and that λ appearing in (49) and (50) is no greater in magnitude than the order of unity.

For an expansion of $f_{kk'}$ on the Fermi surface by (A5), only the terms with $n=0$ and $n=1$ appear to be directly obtainable from experiment, as is well known.[2] It has thus been the habit to assume, as a matter of convenience, that these are the only nonzero terms in (A5). We show that, under such an assumption, $\partial^2 \bar{f}_{kk'}/\partial k \partial k'$ is indentically zero, and hence (25) $f_{kk'}$ makes no contribution to the T^3 term in the specific heat: From (A1) and (A2), $g(\theta)$ is given by

$$g(\theta) = g_0 + \sum_{n=1}^{\infty} g_n (2k_F^2)^{n/2} (1-\cos\theta)^{n/2}. \tag{A8}$$

Comparing (A8) with (A5) shows that if only G_0 and G_1 are nonzero, then only g_0 and g_2 are nonzero. But, from (A3), we have

$$\left(\frac{\partial^2 \bar{g}_{kk'}}{\partial k \partial k'}\right)_{k=k_F, k'=k_F} = \frac{1}{2k_F^2} \sum_{n=1}^{\infty} \frac{g_n}{n+2}$$
$$\times (2k_F)^n (n+1)(n-2), \tag{A9}$$

which is zero if only g_0 and g_2 are nonzero. Hence, within the framework of the Fermi liquid theory and assuming $f_{kk'}$ to depend on $\mathbf{k}$ and $\mathbf{k}'$ only via $|\mathbf{k}-\mathbf{k}'|$, a strong departure of the specific heat from linear behavior at low temperatures implies that the angular dependence of $f_{kk'}$ on the Fermi surface is more complicated than describable by a combination of only $P_0(\cos\theta)$ and $P_1(\cos\theta)$.

APPENDIX B

Effect of Density Dependence upon ϕ

The manner in which ϕ, as computed in Sec. IVA, must be modified when $f_{kk'}$ is density-dependent will now be discussed. To see how this density-dependence arises and its connection with rearrangement energy, it is instructive to write the total energy (neglecting spin dependence) as

$$E = \sum_k \frac{k^2}{2m} n_k + \frac{1}{2} \sum_{kk'} U_{kk'} n_k n_{k'} \tag{B1}$$

where, in Brueckner theory, $U_{kk'}$ is related to a reaction matrix, and is, like E itself, a function of the quasiparticle distribution [Eqs. (6) and (7) of Ref. 22]. The single-particle energy ϵ_k is given (3) by the first functional derivative $\delta E/\delta n_k$ and is thus

$$\epsilon_k = \frac{k^2}{2m} + \sum_{k'} U_{kk'} n_{k'} + \frac{1}{2} \sum_{ll'} \frac{\delta U_{ll'}}{\delta n_k} n_l n_{l'} \tag{B2}$$

with $U_{kk'} = U_{k'k}$. The first two terms in (B.2) represent the single-particle energy as originally defined by Brueckner (called the "model energy" by Thouless[25]) while the final term, which is zero in Hartree-Fock theory, is the rearrangement energy. Landau's $f_{kk'}$ is just (9) the second functional derivative $\delta^2 E/\delta n_k \delta n_{k'}$ multiplied by the volume:

$$V^{-1} f_{kk'} = U_{kk'} + \sum_l n_l \left(\frac{\delta U_{k1}}{\delta n_{k'}} + \frac{\delta U_{k'1}}{\delta n_k} \right)$$
$$+ \frac{1}{2} \sum_{ll'} n_l n_{l'} \frac{\delta^2 U_{ll'}}{\delta n_k \delta n_{k'}}. \tag{B3}$$

If $U_{kk'}$ is a function of the quasiparticle distribution, then $f_{kk'}$ will be as well; and, upon comparing (B2) and (B3), we see that a sufficient condition for $f_{kk'}$ to depend upon the quasiparticle distribution is that the rearrangement energy be nonzero.

In general, therefore, $f_{kk'}$ will be a function both of density and of temperature. Equations (21) and (22) show that the temperature dependence of $f_{kk'}$ may be neglected in a calculation of the T^3 term in the specific heat; so Eq. (25) remains valid. For a computation of terms of order higher than T^3, however, the temperature dependence of $f_{kk'}$ must be included.

To discuss how density dependence of $f_{kk'}$ modifies previous formulas, we adopt, for simplicity, the following model which illustrates the basic points:

$$f_{kk'}(k_F) = h(k_F) F(K), \tag{B4}$$

where $K = |\mathbf{k}-\mathbf{k}'|$ as in (A1). Equation (40) is then seen to become

$$\left(\frac{\partial^2 f_{kk'}}{\partial k \partial k'}\right)_{k=k_F; k'=k_F} = \frac{1}{4}\bar{f}'' - \frac{\bar{f}-f(0)}{2k_F^2} - \frac{1}{4}h''\bar{F} - \frac{1}{2}h'\bar{F}', \tag{B5}$$

where primes are used to indicate differentiation with respect to k_F and $\bar{F}$ is the usual abbreviated notation for the angular average (23) of $F(K)$ over the Fermi surface. Only part of the story, though, is told by (B5) since the relation (38) between $\bar{f}$ and the sound velocity and effective mass is also altered, as shown below.

The dependence of c^2 upon Fermi energy ϵ_{k_F} may be written as[2]

$$mc^2 = \frac{N\pi^2}{Vk_F^2} \frac{d\epsilon_{k_F}}{dk_F}. \tag{B6}$$

[24] F. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1958), 4th ed. p. 302.

[25] D. J. Thouless, Phys. Rev. **112**, 906 (1958).

Equation (9), written in integral form, shows

$$\epsilon_k(k_F) = \frac{k^2 - k_{F_0}^2}{2m_0^*} + \mu_0(k_{F_0}) + \frac{1}{\pi^2} \int_0^\infty \bar{f}_{kk'}[\theta(k_F - k')$$
$$-\theta(k_{F_0} - k')]k'^2 dk', \quad (B7)$$

at zero temperature. Here $\bar{\epsilon}_k$, the single-particle energy at zero temperature and equilibrium density, has been taken to be of the form (7) in which k_{F_0} and m_0^* are the Fermi momentum and effective mass at equilibrium density. At zero temperature, the distribution $n_{k'}$ corresponding to a Fermi momentum k_F is $\theta(k_F - k')$, where $\theta(x)$ is the step function which is zero for $x < 0$ and unity for $x > 0$. For $d\epsilon_{k_F}/dk_F$, we have then

$$\frac{d\epsilon_{k_F}(k_F)}{dk_F} = \frac{k_F}{m_0^*} + \frac{k_F^2}{\pi^2}\bar{f} + \frac{1}{\pi^2}\int_{k_{F_0}}^{k_F} \frac{\partial}{\partial k_F}\bar{f}_{k_Fk'}k'^2 dk'. \quad (B8)$$

But since, from (B7),

$$\frac{d\epsilon_k(k_F)}{dk} = \frac{k}{m_0^*} + \frac{1}{\pi^2}\int_{k_{F_0}}^{k_F} \frac{\partial f_{kk'}}{\partial k}k'^2 dk' \equiv \frac{k}{m^*}, \quad (B9)$$

and, in general,

$$\frac{\partial}{\partial k_F}\bar{f}_{k_Fk'}(k_F) = \left[\frac{\partial \bar{f}_{kk'}(k_F)}{\partial k_F}\right]_{k=k_F} + \left[\frac{\partial \bar{f}_{kk'}(k_F)}{\partial k}\right]_{k=k_F}, \quad (B10)$$

(B8) becomes

$$\frac{d\epsilon_{k_F}}{dk_F} = \frac{k_F}{m^*} + \frac{k_F^2}{\pi^2}\bar{f} + \frac{1}{\pi^2}\int_{k_{F_0}}^{k_F}\left(\frac{\partial \bar{f}_{kk'}}{\partial k_F}\right)_{k=k_F} k'^2 dk'. \quad (B11)$$

Using (B11) in (B6) together with (6) then gives, finally,

$$\left(\frac{V}{N}\right)\left(mc^2 - \frac{k_F^2}{3m^*}\right) = \bar{f} + \frac{1}{k_F^2}\int_{k_{F_0}}^{k_F}\left(\frac{\partial \bar{f}_{kk'}}{\partial k_F}\right)_{k=k_F} k'^2 dk', \quad (B12)$$

which, if $\bar{f}_{kk'}$ is a function of k_F, reduces to (38) only at equilibrium density.

Calling $\bar{f}$ the value of $\bar{f}$ in the absence of density dependence,

$$\bar{f} \equiv (V/N)(mc^2 - k_F^2/3m^*), \quad (B13)$$

and using (B4) and (B5) together with (44a) then yields, after some algebra,

$$\left(\frac{\partial^2 \bar{f}_{kk'}}{\partial k \partial k'}\right)_{k=k_{F_0}, k'=k_{F_0}} = \tfrac{1}{4}\bar{f}'' - \frac{\bar{f} - \bar{f}(0)}{2k_{F_0}^2} - \tfrac{7}{8}h'\bar{f}'$$
$$- \tfrac{1}{4}\bar{f}[3h'' - 7h'^2 - 2h'k_{F_0}^{-1}], \quad (B14)$$

at equilibrium density, where $h(k_{F_0})$ has been taken as unity. Assuming the validity of assumption (B4), the expression (B14) replaces (38) and (40) as the proper relation for computing ϕ. With derivatives of $\bar{f}$ given

by the empirical formula (43), we see that if $h'k_{F_0} \approx \frac{3}{2}$, the third term in (B14) cancels the first, and ϕ is thus very much less than given by (42).

Although there is clearly no justification for an assumption such as (B4), this model serves the fundamental purpose of illustrating how ϕ could be close to an order of magnitude less than estimated in Sec. IVA if, perhaps not unreasonably, $k_F \partial f_{kk'}/\partial k_F$ is the order of $f_{kk'}$.

APPENDIX C

Effect of Density Dependence upon J

If $\zeta_{kk'}$ is a function of Fermi momentum, the situation in regard to the T^2 term in the susceptibility can be quite complicated, for not only will (48) require alteration in a manner similar to (B5) but additional terms will appear in the calculation of Sec. III so that J itself must be redefined. We discuss here, by drawing analogy to the Brueckner-Gammel theory[23] of liquid He[3], how such additional terms arise.

According to Eq. (A3) of Ref. 23, if elements of the reaction matrix are density-dependent, then in computing, for example, the interaction between two particles with spin "up" $(+)$, the proper value of Fermi momentum to use in the reaction matrix is k_F^+, where k_F^+ is the Fermi momentum appropriate to N^+ spins aligned parallel to the external field:

$$(4\pi/3)(k_F^+/2\pi)^3 = N^+/V. \quad (C1)$$

A similar statement is made for interaction between spin "down" particles; and for interaction between antiparallel spins, an average of k_F^+ and k_F^-—which is just k_F itself to first order in H—should be used [Eq. (A2) of Ref. 23]. By direct analogy with this prescription of Brueckner and Gammel, we therefore write (9) as

$$\epsilon_k^\pm = \bar{\epsilon}_k + V^{-1}\sum_{k'} f_{kk'}(k_F^\pm, T)\delta n_{k'}^\pm$$
$$+ V^{-1}\sum_{k'} f_{kk'}(k_F, T)\delta n_{k'}^\mp$$
$$+ \tfrac{1}{4}V^{-1}\sum_{k'} \zeta_{kk'}(k_F^\pm, T)\delta n_{k'}^\pm$$
$$- \tfrac{1}{4}V^{-1}\sum_{k'} \zeta_{kk'}(k_F, T)\delta n_{k'}^\mp \mp \gamma H, \quad (C2)$$

to first order in applied magnetic field H, where $f_{kk'}$ and $\zeta_{kk'}$ are, in general, dependent upon temperature as well as spin density.

Keeping only terms to first order in $k_F^+ - k_F^-$ and second order in T, we have for $\epsilon_k^+ - \epsilon_k^-$:

$$\epsilon_k^+ - \epsilon_k^- = -2\gamma H + (2V)^{-1}\sum_{k'} \zeta_{kk'}(n_{k'}^+ - n_{k'}^-)$$
$$+ V^{-1}(k_F^+ - k_F^-)\sum_{k'}\left(\frac{\partial f_{kk'}}{\partial k_F} + \frac{1}{4}\frac{\partial \zeta_{kk'}}{\partial k_F}\right)\delta n_{k'}$$
$$+ (2V)^{-1}\sum_{k'} \delta\zeta_{kk'}(n_{k'}^+ - n_{k'}^-). \quad (C3)$$

The first two terms in (C3) are the same as in (31); additional terms arise because $f_{kk'}$ and $\zeta_{kk'}$ vary as the medium is polarized, leading to the coefficient of $k_F{}^+$ $-k_F{}^-$, and also because the exchange interaction may be temperature-dependent, as expressed in the final term where $\delta\zeta_{kk'}$ is the change in $\zeta_{kk'}$ to order T^2 in going from zero to a finite temperature. The quantity $\delta n_{k'}$ is, likewise, the change in $n_{k'}$ to order T^2 as the temperature is made finite. Repeating the steps of Sec. III with $\epsilon_k{}^+ - \epsilon_k{}^-$ now defined as in (C3) leads to a new expression for χ with a modified coefficient of T^2. (The zero temperature limit is, of course, unchanged.)

We eliminate $k_F{}^+ - k_F{}^-$ by noting that if

$$N^\pm = \tfrac{1}{2}N(1\pm\nu), \qquad (C4)$$

so that

$$M = \gamma N \nu \qquad (C5)$$

and, to first order in ν,

$$k_F{}^\pm = k_F(1\pm\tfrac{1}{3}\nu) \qquad (C6)$$

then, in terms of χ,

$$k_F{}^+ - k_F{}^- = 2k_F\chi H/3\gamma N. \qquad (C7)$$

Thus, from (C3) and (C7), together with the methods of Sec. III, we have, to order T^2,

$$A_k = \tilde{A}_k - \frac{1}{4}\frac{\chi}{\gamma^2}\left(\frac{T}{T_F{}^*}\right)^2\frac{\pi^2}{12}\frac{k_F{}^2}{V}\left[\frac{\partial}{\partial k'}\left(\frac{\partial \bar{f}_{kk'}}{\partial k_F} + \frac{1}{4}\frac{\partial \bar{\zeta}_{kk'}}{\partial k_F}\right)\right]_{k'=k_F}$$
$$- \frac{3N}{8V}\frac{A_k{}^0}{\kappa T_F{}^*}\delta\bar{\zeta}_{kk_F} \qquad (C8)$$

where A_k is defined in terms of $\epsilon_k{}^+ - \epsilon_k{}^-$ by (28), $\tilde{A}_k$ is

the value of A_k previously computed from (31), and $A_k{}^0$ is the zero-temperature value given by (33).

A final expression for χ is then obtained by substituting (C8) into (29). This may be written as

$$\chi = \frac{3N\gamma^2 A_{k_F}{}^0}{2\kappa T_F{}^*}\left\{1 - \frac{\pi^2}{12}\left(\frac{T}{T_F{}^*}\right)^2\left[A_{k_F}{}^0\left(1+\frac{6}{5}\phi\right) - (J+\Delta J)\right]\right\}, \qquad (C9)$$

similar to (32) where all quantities except ΔJ are as given in Secs. II and III, and ΔJ is the correction due to the density and temperature dependence of $f_{kk'}$ and $\zeta_{kk'}$. Assuming, consistent with the discussion in Sec. VI, that the density dependence of $\zeta_{kk'}$ is much less than that of $f_{kk'}$, only the term in $\partial f_{kk'}/\partial k_F$ is important[26] in (C8). In this approximation,

$$\Delta J = -\frac{3N}{8V}\frac{k_F{}^2}{\kappa T_F{}^*}\left[\frac{\partial}{\partial k'}\left(\frac{\partial f_{kk'}}{\partial k_F}\right)\right]_{k=k_F, k'=k_F}. \qquad (C10)$$

With the assumption (B4) together with other results of Appendix B and the empirical relation (43), we find $\Delta J \approx -4.5$ at equilibrium density and for $k_F\partial f_{kk'}/\partial k_F \approx f_{kk'}$. The effect of this upon values of J previously computed (Table I) is not drastically significant. But it may be noted that since the effect is to decrease J, it makes it all the more unlikely that J can be of the order of 10^2, as needed to obtain consistency with the Illinois specific-heat measurements.[4]

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[26] If the density-dependence of $\zeta_{kk'}$ is negligible, then so is its temperature dependence since both effects are caused by $\zeta_{kk'}$ being a function of the quasiparticle distribution.